these trajectories. In particular, the measurement of high-energy pion-nucleon charge exchange should be encouraged. Because of G-parity conservation, the R trajectory would be absent here and, if the energy is high enough, the ρ alone should suffice.

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Determination of Pion-Nucleon S-Wave Scattering Lengths by the N/D Method

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The pion-nucleon S-wave scattering lengths are calculated using the method originally developed by Balázs, wherein an effective-range two-pole approximation is made for the numerator function. The residues of the effective-range poles are determined by matching the amplitude and its derivative with those calculated with a fixed energy dispersion relation. In calculating the latter the contribution of only the N^* and ρ, together with those of appropriate nucleon-pole terms, are retained. The calculated scattering length in the $T=\frac{3}{2}$ state is in excellent agreement with experimental result, while for the $T=\frac{1}{2}$ state, the calculated value, though of the right sign, is about twice the experimental scattering length.

SEVERAL approximation schemes[1–4] have recently been suggested to calculate the elements of the S matrix of strongly interacting particles. These attempts aim at constructing approximate solutions for the scattering amplitude consistent with the requirements of analyticity, elastic unitarity, and crossing symmetry. Among the methods, the one suggested by Balázs in which an effective-range approximation is made to represent the effect of the distant crossed-channel singularities, has the advantage of being free from the necessity of introducing arbitrary parameters into the theory. The Balázs method has been applied to the pion-nucleon problem by Singh and Udgaonkar[5] who have made a self-consistent calculation of the mass and width of the pion-nucleon (3,3) isobar, N^*. The present investigation which may be considered as a continuation of the work of these authors, is devoted to the study of the pion-nucleon S-wave amplitude using the aforementioned techniques. Our procedure is as follows: We use the N/D equations, and represent the N function by a two-pole effective-range formula. The residues of these poles, whose positions have been fixed *a priori*, are next evaluated by comparing the amplitude and its derivative at a suitably chosen point, with the values of the same quantities calculated with the help of a fixed energy dispersion relation. In calculating the latter only the contributions of N^* and ρ, together with those of the appropriate nucleon-pole terms are retained. In this way the partial-wave amplitude is completely determined and the S-wave scattering lengths may now be readily determined. The calculated scattering length in the $T=\frac{3}{2}$ state comes out to be in excellent agreement with the experimental result, while for the $T=\frac{1}{2}$ state the calculated scattering length, although of the right order and having the correct sign, is much too large. This may be due to our explicit neglect of the inelastic channels which are expected to be relatively more important in the $T=\frac{1}{2}$ state.

We follow the same notation as in Frautschi and Walecka.[6] Let us consider the S-wave amplitude f_{0+} normalized as

$$f_{0+}=\frac{W^2}{q}e^{i\delta}\sin\delta \tag{1}$$

and write it in the N/D form

$$f_{0+}=N(s)/D(s). \tag{2}$$

In (1), $W^2(=S)$ is the square of the total c.m. energy of the incoming particles and q the magnitude of c.m. 3-momentum. δ is the $S_{1/2}$ phase shift. In the two-pole effective-range approximation the $N(s)$ function may be written as[7]

$$N(s)=\frac{R_1}{s+m^2}+\frac{R_2}{s+16m^2}. \tag{3}$$

In (3), m is the nucleon mass. The pion mass has been

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[6] S. C. Frautschi and J. D. Walecka, Phys. Rev. **120**, 1486 (1960).
[7] This is shown in Ref. 5. The results of this calculation do not appreciably depend on the variation of the location of effective-range poles [B. M. Udgaonkar (private communication)]. This may also be true in the present case.

set equal to unity. The $D(s)$ function is now given by

$$D(s)=1-\frac{s-(m-1)^2}{\pi}\int_{(m+1)^2}^{\infty}ds'\frac{q'/s'}{(s'-s)(s'-(m-1)^2)}\times\left\{\frac{R_1}{s'+m^2}+\frac{R_2}{s'+16m^2}\right\}. \quad (4)$$

The unknown residues R_1 and R_2 will now be determined by matching the amplitude f_{0+} given by (3) and (4), and its derivative, with the values calculated with a fixed s dispersion relation. The fixed s dispersion relation satisfied by the invariant amplitude $A(s,t)$ is

$$A(s,t,u)=\text{Pole terms}+\int_{(m+1)^2}^{\infty}\frac{A_u(u')}{u'-u}du'+\int_4^{\infty}\frac{A_t(t')}{t'-t}dt'. \quad (5)$$

A similar relation holds for $B(s,t,u)$. s, t, u are the usual Mandelstam variables. Using (5), we calculate $A(s,t)$ by assuming that the u' and t' integrals are exhausted by the contributions of the N^* and ρ, respectively. $B(s,t)$ is similarly calculated. With the expressions for $A(s,t)$ and $B(s,t)$ so calculated one can easily calculate the partial-wave amplitude $f_{0+}(s)$. This has the form

$$f_{0+}(s)=f_{0+}^{(N)}(s)+f_{0+}^{(\rho)}(s)+f_{0+}^{(N^*)}(s). \quad (6)$$

$f_{0+}^{(N)}(s)$, $f_{0+}^{(\rho)}(s)$, and $f_{0+}^{(N^*)}(s)$ denote the contributions of the nucleon, the ρ meson and N^*, respectively. These contributions are now written down for the $T=\frac{1}{2}$ state. The contributions of ρ is given by

$$f_{0+}^{(\rho)}(s)=\frac{1}{32\pi}\{[(s^{1/2}+m)^2-1][A_0^{(\rho)}+(s^{1/2}-m)B_0^{(\rho)}]+[(s^{1/2}-m)^2-1][-A_1^{(\rho)}+(s^{1/2}+m)B_1^{(\rho)}], \quad (7)$$

with $A_{0,1}^{(\rho)}$, $B_{0,1}^{(\rho)}$ given by

$$A_{0,1}^{(\rho)}=-\frac{12\pi\gamma_2}{m}\frac{(2m^2+2-m_\rho^2-2s)}{[s-(m+1)^2]}\frac{4s}{[s-(m-1)^2]}\times Q_{0,1}\left(1+\frac{2m_\rho^2 s}{[s-(m+1)^2][s-(m-1)^2]}\right), \quad (8)$$

$$B_{0,1}^{(\rho)}=-\frac{24\pi(\gamma_1+2\gamma_2)-4s}{[s-(m+1)^2][s-(m-1)^2]}\times Q_{0,1}\left(1+\frac{2sm_\rho^2}{[s-(m+1)^2][s-(m-1)^2]}\right). \quad (9)$$

In the above, m_ρ is the mass of the ρ meson, and $Q_l(x)$ is the Legendre function of the second kind. The parameters γ_1 and γ_2 which occur in the above expressions may be determined from an analysis of the electromagnetic form factor of the nucleon. We take the results of a recent investigation of this problem by Singh and Udgaonkar[8] which gives

$$\gamma_1=-4.91 \quad \text{and} \quad \gamma_2=-11.7.$$

The N^* contribution $f_{0+}^{(N^*)}$ is given by an expression similar to (7) with $A_{0,1}^{(\rho)}$ and $B_{0,1}^{(\rho)}$ replaced by $A_{0,1}^{(N^*)}$ and $B_{0,1}^{(N^*)}$. These are given by

$$A_{0,1}^{(N^*)}(s)=\frac{4sX}{[s-(m+1)^2][s-(m-1)^2]}\times Q_{0,1}\left(1+\frac{2s(2m^2+2-m_3^2-s)}{[s-(m+1)^2][s-(m-1)^2]}\right), \quad (10)$$

$$B_{0,1}^{(N^*)}(s)=\frac{4sY}{[s-(m+1)^2][s-(m-1)^2]}\times Q_{0,1}\left(1+\frac{2s(2m^2+2-m_3^2-s)}{[s-(m+1)^2][s-(m-1)^2]}\right). \quad (11)$$

X and Y occurring above are given by the following expressions:

$$X=\frac{32\pi\gamma_3}{3}\left[(m_3+m)\frac{2m^2-m_3^2-s}{2}+\frac{2(m_3+m)}{3m_3^2}\left(\frac{m_3^2-m^2+1}{2}\right)^2+\frac{m_3+m}{3}(m_3^2-m^2)-\frac{(m_3^2-m^2)^2}{6m_3}\right], \quad (12)$$

$$Y=-\frac{32\pi\gamma_3}{3}\left[\frac{2m^2-m_3^2-s}{2}+\frac{2}{3m_3^2}\left(\frac{m_3^2-m^2+1}{2}\right)^2-\frac{2m}{3}(m_3+m)+\frac{m}{m_3}(m_3^2-m^2+1)+\frac{1}{3}\right]. \quad (13)$$

In the above, $m_3(=8.8)$ and $\gamma_3(=0.06)$ are, respectively, the mass and the reduced half-width of the N^*. In writing down Eqs. (7)–(13) we have also followed the usual practice of neglecting the width of a resonance (compared to its mass) in the energy denominator. This procedure is rigorously valid only for the case of a stable particle, however. Finally let us consider the nucleon

[8] V. Singh and B. M. Udgaonkar, Phys. Rev. **128**, 1820 (1962).

contribution. This is given by

$$f_{0+}^{(N)}(s)=-\frac{3}{16\pi}g^2\left\{\frac{[(s^{1/2}+m)^2-1][s^{1/2}-m]}{s-m^2}\right\}+\frac{1}{32\pi}$$
$$\times\{[(s^{1/2}+m)^2-1](s^{1/2}-m)B_0^{(N)}$$
$$+[(s^{1/2}-m)^2-1](s^{1/2}+m)B_1^{(N)}\}. \quad (14)$$

$B_0^{(N)}$ and $B_1^{(N)}$ are given by

$$B_{0,1}^{(N)}(s)=-\frac{4g^2s}{[s-(m+1)^2][s-(m-1)^2]}$$
$$\times Q_{0,1}\left(1+\frac{2s(m^2+2-s)}{[s-(m+1)^2][s-(m-1)^2]}\right). \quad (15)$$

g^2 occurring above is the rationalized pion-nucleon coupling constant, $g^2/4\pi\simeq 14$. The first term in (14) is the nucleon-pole term in the direct s channel, and the second term that in the u channel. Using the said matching procedure, the partial-wave amplitude f_{0+} is now completely determined. We take the matching point at $s=(m-1)^2$. From the partial-wave amplitude f_{0+}, so determined, the scattering length which is defined as

$$\frac{s}{(1/a)-iq}=f_{0+}(s), \quad a=\frac{f_{0+}(s=(m+1)^2)}{(m+1)^2}, \quad (16)$$

may be easily obtained. The calculated value of the $T=\frac{1}{2}$ scattering length is

$$a_1\simeq 0.31,$$

which is to be compared with the value obtained from experimental data by Woolcock[9]

$$a_1=0.17\pm 0.005.$$

A similar treatment may be done for the $T=\frac{3}{2}$ state. The contribution of the various states to the fixed s-dispersion relation for this case may be obtained from the corresponding values for $T=\frac{1}{2}$ states by the relation

$$f_{0+,3/2}^{(N^*)}=\tfrac{1}{4}f_{0+,1/2}^{(N^*)}; \quad f_{0+,3/2}^{(\rho)}=-\tfrac{1}{2}f_{0+,1/2}^{(\rho)};$$
$$f_{0+,3/2}^{(N)}=-2f_{0+,1/2}^{(N)}. \quad (17)$$

The substitution rule for $f^{(N)}$ refers to the u-channel nucleon only, there being no nucleon pole in s channel for the $T=\frac{3}{2}$ state. The calculated $T=\frac{3}{2}$ scattering length is

$$a_3\simeq -0.088,$$

which is to be compared to the experimental result[9]

$$a_3=-0.089\pm 0.004.$$

The calculated scattering lengths are of the right order and have correct signs. For the $T=\frac{3}{2}$ state the agreement between the theoretical and experimental results is extremely close. This may be partially accidental. For the $T=\frac{1}{2}$ state the calculated value is about twice the experimental result. There may be several reasons for such a situation. First is the importance of inelastic channels which have not been taken into account in our calculation. For the $T=\frac{1}{2}$ state one expects them to be relatively important because of the existence of two-body inelastic channels such as $(N\eta)$, (N,ABC), $(K\Lambda)$, etc., fairly close to the threshold. The second reason may be our inability to treat the high-energy parts of the crossed channels occurring in the fixed s-dispersion relation properly. The only tractable method of taking the high-energy effects into consideration is the Singh-Udgaonkar[10] approximation, which relates these effects to the low-energy resonances in the direct channel. Within the restricted criterion of the validity of this approximation, however, the high-energy effects for the present case can only arise from the contributions of higher spin resonances in the direct channel; there being no low-energy direct channel resonance in the $S_{1/2}$ partial wave. In particular, the $T=\frac{1}{2}$ state can have a contribution from the $T=\frac{1}{2}$, $d_{3/2}$ resonance at 600 MeV in the pion-nucleon system. Our neglect of this contribution might be partly responsible for the discrepancy in the $T=\frac{1}{2}$ scattering length.

Previous workers[11] found it difficult to explain the isotopic spin dependence of the S-wave scattering length without making arbitrary assumptions.[12,13] Our success in explaining this effect may be considered as the main result of this investigation. We therefore conclude that the approximation scheme (Ref. 5) which has been successfully applied to the study of pion-nucleon (3,3) partial wave, together with the same choice for the location of the effective-range poles and the same matching point, also explains the qualitative features of low-energy scattering in the $S_{1/2}$ state.

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[9] W. S. Woolcock, *Proceedings of the Aix-en-Provence International Conference on Elementary Particles, 1961* (Centre d'Etudes Nucléaires de Saclay, Seine et Oise, 1961), p. 459.

[10] See Ref. 5. A Reggeized version of this approximation has been proposed by L. A. P. Balázs, Phys. Rev. **132**, 867 (1963).

[11] J. L. Uretsky, Phys. Rev. **123**, 1459 (1961).

[12] J. J. Sakurai, Proceedings of International School of Physics, Varenna, 1963 (unpublished). For instance, the validity of Sakurai's observation that a simple ρ-exchange diagram explains the isotopic spin dependence of the scattering length depends crucially on the explicit neglect of the magnetic moment coupling (the parameter γ_2 used in text) of the ρ meson.

[13] J. Bowcock, W. N. Cottingham, and D. Lurie, Nuovo Cimento **16**, 918 (1960).